S' = stoichiometric matrix formed by deleting dependent columns in **M**

S'' = submatrix of S' corresponding to Bodenstein products

x_j = molar extent of reaction to j

x = vector of x_j values

Z = matrix the columns of which constitute a basis of the null space of S''

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Analysis of Heat Transfer for Laminar Power Law Pseudoplastic Fluids in a Tube With an Arbitrary Circumferential Wall Heat Flux

An analysis is performed to obtain an exact solution to the problem of thermal entry region heat transfer in a circular tube with an arbitrary circumferential wall heat flux for pseudoplastic fluids using the power law constitutive model. The solution is expanded in a power series form, with expansion coefficients and related constants obtained numerically. A simple result is presented for a cosine heat flux distribution around the periphery of the tube which illustrates the simultaneous influence of circumferential wall heat flux variation and non-Newtonian fluid behavior on heat transfer.

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SCOPE

The problem of laminar, forced convection flow of a fluid through a pipe with a heated wall is classic in the heat transfer literature. Analytical and experimental studies have been reported with various flow and wall conditions for both Newtonian and non-Newtonian fluids.

The motivation for the present work arose from the

consideration of non-Newtonian fluids flowing in pipes with wall heating which is circumferentially variable. This would occur, for instance, in a furnace where tubes carrying the fluid are adjacent to the combustion zone. Examination of the behavior of non-Newtonian fluids in this case is of general interest; Newtonian behavior is a special case of whatever constitutive equation is used to model non-Newtonian behavior.

The specific conditions examined here are laminar flow in a circular conduit with wall heating which is constant axially but variable circumferentially. The velocity profile is presumed parabolic, that is, fully developed, throughout. However, both the fully developed and entry region are considered insofar as temperature profiles are concerned. A power law form is used to model non-Newtonian fluid behavior. Circumferential variation of wall heat flux is constrained only in that it must be expressible in terms of a Fourier expansion.

Analytical results were obtained for these conditions with a final solution requiring input of the wall heat flux $q(\phi)$. The general solution is shown to reduce to the special cases for which known solutions exist, namely, asymptotic (fully developed temperature profile) case, the case of a Newtonian fluid with developing and fully developed temperature profiles, and with non-Newtonian fluids subjected to a wall heat flux which is uniform both axially and circumferentially.

CONCLUSIONS AND SIGNIFICANCE

A complete solution is reported for laminar, fully developed flow in a circular pipe with uniform entrance temperature subjected to a wall heat flux which is uniform axially but circumferentially variable. Results are presented for non-Newtonian fluid behavior characterized by a power law constitutive relationship.

The complete solution is expressed, in nondimensional form, as

$$\theta(x^+, r^+, \phi) = \theta_{fd}(x^+, r^+, \phi) + \theta^+(x^+, r^+, \phi)$$

where θ_{fd} is the asymptotic temperature distribution, and θ^+ is the entry region solution. Expressions for wall temperature distribution and local Nusselt number are presented; these are easily obtained from the complete solution for θ.

The expressions generated for fluid temperature, wall temperature, and local Nusselt number are given in terms of the wall heat flux distribution $q(\phi)$. Any circumferential distribution is allowed so long as it may be expressed in Fourier series form.

The complete solution is shown to reduce to known forms for the special cases with $q(\phi)$ given as $q(\phi) = q_{av}(1 + \cos\phi)$. Cases examined are a fully developed temperature profile (asymptotic solution) with non-Newtonian fluids, the entry length solution with Newtonian fluids, and non-Newtonian fluids with uniform wall heat flux. A complete solution is illustrated where both non-Newtonian fluid behavior as well as circumferentially varying wall heat flux are present. The eigenvalues, eigenfunctions, and expansion coefficients which apply have been calculated and are presented both in tabular and graphical forms. Plots of wall temperature and local Nusselt number are also given for this case.

Literature in the area of non-Newtonian tube flow with an arbitrary variation of circumferential wall heat flux or wall temperature is relatively sparse. The work of Inman (1965), who considered asymptotic velocity and temperature profiles, is singular in this regard.

This investigation extends the work of Inman to solve the problem of heat transfer in a circular tube with an arbitrary circumferential wall heat flux for the case of a developing temperature profile with power law, pseudoplastic fluids.

The solution was expanded in a power series form that accounts for any arbitrary variation of wall heat flux around the circumference that can be expressed in terms of a Fourier expansion. The first twelve eigenvalues, eigenfunctions, and the expansion coefficients were obtained numerically.

For the limiting case of power law, pseudoplastic fluids with uniform wall heat flux, the eigenfunctions, eigenvalues, and expansion coefficients reduce to the values reported by Michiyoshi and Matsumoto (1964) and Mitsuishi and Miyatake (1967). The problem of Newtonian fluids with an arbitrary circumferential wall heat flux is another limiting case of the present work. The related constants for this problem are the same as those obtained by Battacharya and Roy (1970).

Finally, a simple result has been obtained for a cosine heat flux around the tube periphery which illustrates all of the limiting cases and shows the simultaneous influences of circumferential wall heat flux variation and non-Newtonian fluid behavior on heat transfer.

FORMULATION OF PROBLEM

The problem to be treated is represented schematically in Figure 1. We consider a steady, hydrodynamically developed flow of laminar, constant property, nondissipative viscous fluid flowing in a circular tube. The wall heat flux varies circumferentially according to a general function $q(\phi)$ that can be expressed in terms of a Fourier expansion.

The applicable form of the energy equation is

$$\frac{u(r)}{\alpha}\frac{\partial t}{\partial x} = \frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2} \tag{1}$$

For power law fluids, the velocity distribution is given by Rohsenow (1964):

$$\frac{u}{v} = \frac{3n+1}{n+1}\left[1 - \left(\frac{r}{r_0}\right)^{(n+1)/n}\right] \tag{2}$$

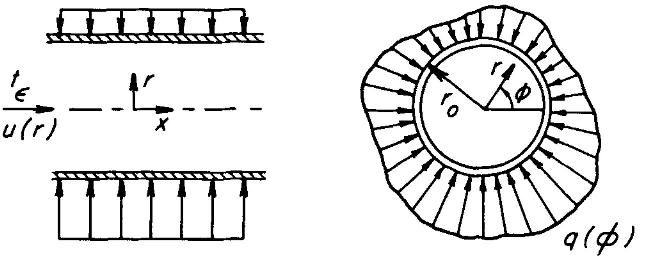

Fig. 1. Physical model and coordinate system.

For convenience, we define

$$s = \frac{n+1}{n}$$

and rewrite Equation (2) as

$$\frac{u}{v} = \frac{s+2}{s}\left[1 - \left(\frac{r}{r_O}\right)^s\right] \quad (3)$$

Note if $s = 2$ or $n = 1$, the fluid is Newtonian; if $s = \infty$ or $n = 0$, plug flow is obtained, and $s = 1$ or $n = \infty$ is the limiting case of dilatant fluids.

With $u(r)$ given by Equation (3), the energy equation becomes

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2} = \frac{u_{\max}}{\alpha}\left[1 - \left(\frac{r}{r_O}\right)^s\right]\frac{\partial t}{\partial x} \quad (4)$$

subject to the boundary conditions

$$t(0, r, \phi) = t_\epsilon \quad (5a)$$

$$k\frac{\partial t}{\partial r}(x, r_O, \phi) = q(\phi) \quad (5b)$$

$$t(x, 0, \phi) = \text{finite} \quad (5c)$$

$$t(x, r, \phi) = t(x, r, \phi + 2\pi) \quad (5d)$$

$$\frac{\partial t}{\partial \phi}(x, r, \phi) = \frac{\partial t}{\partial \phi}(x, r, \phi + 2\pi) \quad (5e)$$

Equations (4) and (5) can be expressed in terms of the following dimensionless variables:

$$\theta = \frac{t - t_\epsilon}{\bar{q}\, 2r_O/\pi k} \quad (6a)$$

$$r^+ = \frac{r}{r_O} \quad (6b)$$

$$x^+ = \frac{2s}{s+2}\frac{x/r_O}{RePr} = \frac{2v}{u_{\max}}\frac{x/r_O}{RePr} \quad (6c)$$

where

$$\bar{q} = \int_0^{2\pi} q(\phi)d\phi \quad (6d)$$

with the requirement that $\bar{q} \neq 0$. Performing the necessary transformations, we obtain the dimensionless form of the energy equation

$$\frac{\partial^2 \theta}{\partial r^{+2}} + \frac{1}{r^+}\frac{\partial \theta}{\partial r^+} + \frac{1}{r^{+2}}\frac{\partial^2 \theta}{\partial \phi^2} = (1 - r^{+s})\frac{\partial \theta}{\partial x^+} \quad (7a)$$

and the boundary conditions become

$$\theta(0, r^+, \phi) = 0 \quad (7b)$$

$$\frac{\partial \theta}{\partial r^+}(x^+, 1, \phi) = \frac{q(\phi)}{\bar{q}}\frac{\pi}{2} \quad (7c)$$

$$\theta(x^+, 0, \phi) = \text{finite} \quad (7d)$$

$$\theta(x^+, r^+, \phi) = \theta(x^+, r^+, \phi + 2\pi) \quad (7e)$$

$$\frac{\partial \theta}{\partial \phi}(x^+, r^+, \phi) = \frac{\partial \theta}{\partial \phi}(x^+, r^+, \phi + 2\pi) \quad (7f)$$

Because of experience with heat conduction problems of similar form, a solution is sought having the form

$$\theta(x^+, r^+, \phi) = \theta_{fd}(x^+, r^+, \phi) + \theta^+(x^+, r^+, \phi) \quad (8)$$

in which $\theta_{fd}(x^+, r^+, \phi)$ is the asymptotic solution obtained far downstream where the temperature profile is fully developed, and θ^+ is the entry region solution.

Combining Equations (7) and (8), we obtain two differential equations and associated boundary conditions for the two regions as follows:

$$\frac{\partial^2 \theta_{fd}}{\partial r^{+2}} + \frac{1}{r^+}\frac{\partial \theta_{fd}}{\partial r^+} + \frac{1}{r^{+2}}\frac{\partial^2 \theta_{fd}}{\partial \phi^2} = (1 - r^{+s})\frac{\partial \theta_{fd}}{\partial x^+} \quad (9a)$$

$$\frac{\partial \theta_{fd}}{\partial r^+}(x^+, 1, \phi) = \frac{q(\phi)}{\bar{q}}\frac{\pi}{2} \quad (9b)$$

$$\theta_{fd}(x^+, 0, \phi) = \text{finite} \quad (9c)$$

$$\theta_{fd}(x^+, r^+, \phi) = \theta_{fd}(x^+, r^+, \phi + 2\pi) \quad (9d)$$

$$\frac{\partial \theta_{fd}}{\partial \phi}(x^+, r^+, \phi) = \frac{\partial \theta_{fd}}{\partial \phi}(x^+, r^+, \phi + 2\pi) \quad (9e)$$

$$\frac{\partial^2 \theta^+}{\partial r^{+2}} + \frac{1}{r^+}\frac{\partial \theta^+}{\partial r^+} + \frac{1}{r^{+2}}\frac{\partial^2 \theta^+}{\partial \phi^2} = (1 - r^{+s})\frac{\partial \theta^+}{\partial x^+} \quad (10a)$$

$$\theta^+(0, r^+, \phi) = -\theta_{fd}(0, r^+, \phi) \quad (10b)$$

$$\theta^+(x^+, 0, \phi) = \text{finite} \quad (10c)$$

$$\frac{\partial \theta^+}{\partial r^+}(x^+, 1, \phi) = 0 \quad (10d)$$

$$\theta^+(x^+, r^+, \phi) = \theta^+(x^+, r^+, \phi + 2\pi) \quad (10e)$$

$$\frac{\partial \theta^+}{\partial \phi}(x^+, r^+, \phi) = \frac{\partial \theta^+}{\partial \phi}(x^+, r^+, \phi + 2\pi) \quad (10f)$$

DISCUSSION OF SOLUTION

Equations (9) are satisfied by a solution of the form

$$\theta_{fd} = \frac{-s^2 - 6s - 12}{16(s+2)(s+4)} + \frac{x/r_O}{RePr} + \frac{s+2}{8s}r^{+2} - \frac{r^{+s+2}}{2(s+2)s} + \sum_{n=1}^{\infty} r^{+n}(a_n \cos n\phi + b_n \sin n\phi) \quad (11)$$

where the coefficients a_n and b_n may be determined from boundary condition (9b) with the usual Fourier analysis:

$$\left\{\begin{array}{c} a_n \\ b_n \end{array}\right\} = \frac{1}{2n}\int_0^{2\pi} \frac{q(\phi)}{\bar{q}}\left\{\begin{array}{c} \cos n\phi \\ \sin n\phi \end{array}\right\} d\phi \quad (12)$$

Equations (10), applying to the entry region, are satisfied by

$$\theta^+(x^+, r^+, \phi) = \sum_{n=1}^{\infty}\sum_{p=0}^{\infty} e^{-\lambda^2_{np}x^+} R_{np}(r^+) [a_{np}\cos p\phi + b_{np}\sin p\phi] \quad (13)$$

where λ_{np} and R_{np} are the eigenvalues and eigenfunctions determined from the characteristic equation

$$\frac{1}{r^+}\frac{d}{dr^+}\left[r^+\frac{dR_{np}}{dr^+}\right] + \left[\lambda^2_{np}(1 - r^{+s}) - \frac{p^2}{r^{+2}}\right]R_{np} = 0 \quad (14)$$

with

$$\frac{dR_{np}}{dr^+}(1) = 0; \quad R_{np}(0) = \text{finite}$$

Coefficients a_{np} and b_{np} in Equation (13) are obtained using condition (10b) together with the usual Fourier analysis. The expressions which result are

$$a_{nO} = \frac{\int_0^1 r^+(1-r^{+s})\left(\frac{s^2+6s+12}{16(s+2)(s+4)} - \frac{s+2}{8s}r^{+2} + \frac{(r^+)^{s+2}}{2(s+2)s}\right) R_{nO}(r^+)dr^+}{\int_0^1 r^+(1-r^{+s})R^2_{nO}(r^+)dr^+} \quad (15a)$$

$$\begin{Bmatrix} a_{np} \\ b_{np} \end{Bmatrix} = \frac{1}{\pi}\frac{\int_0^1\int_0^{2\pi} r^+(1-r^{+s})\,\hat{\theta}_{fd}\begin{Bmatrix} \cos p\phi \\ \sin p\phi \end{Bmatrix} R_{np}(r^+)d\phi\, dr^+}{\int_0^1 r^+(1-r^{+s})\,R^2_{np}(r^+)dr^+} \quad (15b)$$

where $\hat{\theta}_{fd}$ is given by

$$\hat{\theta}_{fd} = \frac{s^2+6s+12}{16(s+2)(s+4)} - \frac{s+2}{8s}r^{+2} + \frac{(r^+)^{s+2}}{2(s+2)s} - \sum_{n=1}^{\infty} r^{+n}\,(a_n \cos n\phi + b_n \sin n\phi) \quad (16)$$

At this point, the complete solution is obtained by adding the fully developed solution, Equation (11), to the solution for the thermal entrance region, Equation (13).

The mean fluid temperature over a given cross section may be determined by usual procedures; the expression which results is

$$\theta_m(x^+) = \frac{x/r_O}{RePr}$$

Of particular practical interest are the values of wall temperature and local Nusselt number. The wall temperature is obtained by evaluating the complete solution at $r^+ = 1$, yielding

$$\frac{t_w - t_m}{\overline{q}\;2r_O/k\pi} = \frac{s^2+10s+20}{16(s+2)(s+4)} + \sum_{n=1}^{\infty}(a_n \cos n\phi + b_n \sin n\phi) + \sum_{n=1}^{\infty} a_{nO}R_{nO}(1)e^{-\lambda^2_{nO}x^+} + \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} e^{-\lambda^2_{np}x^+}R_{np}(1)\,(a_{np}\cos p\phi + b_{np}\sin p\phi) \quad (17)$$

Local Nusselt number is now determined from this expression to yield

$$Nu(x^+,\phi) = \pi\frac{q(\phi)}{\overline{q}}\left[\frac{s^2+10s+20}{16(s+2)(s+4)} + \sum_{n=1}^{\infty}(a_n \cos n\phi + b_n \sin n\phi) + \sum_{n=1}^{\infty} a_{nO}R_{nO}(1)e^{-\lambda^2_{nO}x^+} + \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} e^{-\lambda^2_{np}x^+}R_{np}(1)\right.$$

$$\left.(a_{np}\cos p\phi + b_{np}\sin p\phi)\right]^{-1} \quad (18)$$

SPECIAL EXAMPLES

As an illustrative case where this method yields useful information, a cosine circumferential heat flux distribution of the form $q(\phi) = q_{av}(1 + b\cos\phi)$ is considered. Four special examples will be mentioned here. The interested reader may refer to Faghri (1974) for details of those solutions, including tables of eigenvalues and eigenfunctions.

Case 1. Letting $x^+ \to \infty$, Equation (18) reduces to the asymptotic solution which is identical to that presented by Inman (1965).

Case 2. With x^+ finite and $s = 2$ (Newtonian fluids), the Nusselt number expression reduces to the form given by Battacharya and Roy (1970).

Case 3. For uniform wall heat flux ($b = 0$) and pseudo plastic fluids ($s > 2$), the coefficients are summarized in Table 1 and compared with Michiyoshi and Matsumoto (1964) and Mitsuishi and Miyatake (1967).

Case 4. We now wish to investigate the simultaneous effects of both circumferential wall heat flux variation and non-Newtonian fluid behavior on heat transfer.

Solving for a_n, b_n, and $\overline{q}$ using Equations (12) and (6d) and subsequently substituting into Equations (17) and (18), we obtain the following results for wall temperature and for local Nusselt numbers:

$$\frac{t_w - t_m}{q_{av}r_O/k} = \frac{s^2+10s+20}{4(s+2)(s+4)} + \sum_{n=1}^{\infty}\hat{a}_{nO}R_{nO}(1)e^{-\lambda^2_{nO}x^+} + b\cos\phi\left[1 + \sum_{n=1}^{\infty}\hat{a}_{n1}R_{n1}(1)e^{-\lambda^2_{n1}x^+}\right] \quad (19)$$

$$Nu(\phi, x^+) = 2(1 + b\cos\phi)\left\{\frac{2+10s+20}{4(s+2)(s+4)} + \sum_{n=1}^{\infty}\hat{a}_{nO}R_{nO}(1)e^{-\lambda^2_{nO}x^+} + b\cos\phi\left[1 + \sum_{n=1}^{\infty}\hat{a}_{n1}R_{n1}(1)e^{-\lambda^2_{n1}x^+}\right]\right\}^{-1} \quad (20)$$

The coefficients $\hat{a}_{nO}$ and $\hat{a}_{n1}$ are obtained from Equations (15) and (16), with further simplification achieved by use of the characteristic Equation (14) which yields the expression

$$\hat{a}_{np} = \frac{-R_{np}(1)}{\lambda^2_{np}\int_0^1 r^+(1-r^{+s})\,R^2_{np}(r^+)dr^+} \quad (21)$$

with $p = 0, 1, 2, 3 \ldots$

The first twelve eigenvalues, eigenfunctions, and the expansion coefficients have been evaluated for several values of the non-Newtonian fluid behavior index s and for $p = 0, 1, 2, 3, 4, 5$. These are reported by Faghri (1974). The eigenvalues and eigenfunctions were obtained from a numerical solution of Equation (14); the expansion coefficients were found from Equation (21).

Table 1. Comparison of Eigenvalues, Eigenfunctions, and Expansion Coefficients for Pseudoplastic Fluids and Uniform Wall Heat Flux with the Results of Michiyoshi and Matsumoto (1964) and Mitsuishi and Miyatake (1967)

	Michiyoshi and Matsumato (1964)			Mitsuishi and Miyatake (1967)			Present work		
n	λ^2_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$	λ^2_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$	λ_{n0}	$R_{n0}(1)$	$\hat{a}_{n0}$
$s = 4$									
1	20.7623	−0.4594		20.75621	−0.459361	0.374948	4.5555898	−0.4593614	0.3749484
2	67.7523	0.3684		67.67724	0.368174	−0.162983	8.2266127	0.3681742	−0.1629858
3	140.8654	0.3219		140.5581	−0.321533	0.098167	11.8557713	−0.3215268	0.0981749
4	240.1917	0.2923					15.4706322	0.2916657	−0.0680736
5	366.0957	−0.2667					19.0787579	−0.2703117	0.0510894
6							22.6831188	0.2539907	−0.0403443
7							26.2851386	−0.2409491	0.0330107
8			Not				29.8855920	0.2301905	−0.0277273
9			presented				33.4849398	−0.2211002	0.0237634
10							37.0834748	0.2123750	−0.0206940
11							40.6813927	−0.2064372	0.0182562
12							44.2788297	0.2003889	−0.0162796
$s = 6$									
1	18.9927	−0.4400		18.98420	−0.439976	0.362294	4.3570857	−0.4399761	0.3622953
2	62.2528	0.3521		62.183377	0.351922	−0.156065	7.8856682	0.3519216	−0.1560682
3	129.5550	−0.3072		129.2773	−0.306845	0.093787	11.3699979	−0.3068454	0.0937854
4	220 9273	0.2782					14.8399911	0.2780651	−0.0649429
5	338.4745	−0.2683					18.3032563	−0.2575228	0.0486964
6							21.7627592	0.2418446	−0 0384298
7							25.2199189	−0.2293306	0.0314287
8			Not				28.6755081	0.2190168	−0.0263883
9			presented				32.1299868	−0.2103090	0.0226086
10							35.5836478	0.2028178	−0.0196830
11							39.0366869	−0.1962756	0.0173605
12							42.4892407	0.1904915	−0.0154778

Variation in the first two eigenfunctions are shown in Figures 2 and 3 for several values of the non-Newtonian parameter s.

Having evaluated eigenvalues, eigenfunctions, and expansion coefficients, we are now able to determine values for temperature and the Nusselt number at locations of interest. For the case of a Newtonian fluid ($s = 2$) and uniform wall heat flux ($b = 0$), a comparison of these results with Kay's (1966) is given in Table 2. Table 3 presents the local Nusselt number for five different values of the non-Newtonian fluid behavior index. Dimensionless wall-to-bulk temperature difference has been plotted in Figure 4 as a function of angular position ϕ at $x^+ = 0.01$. The effect of non-Newtonian behavior on peripheral wall temperature variation becomes less pronounced far from the entrance.

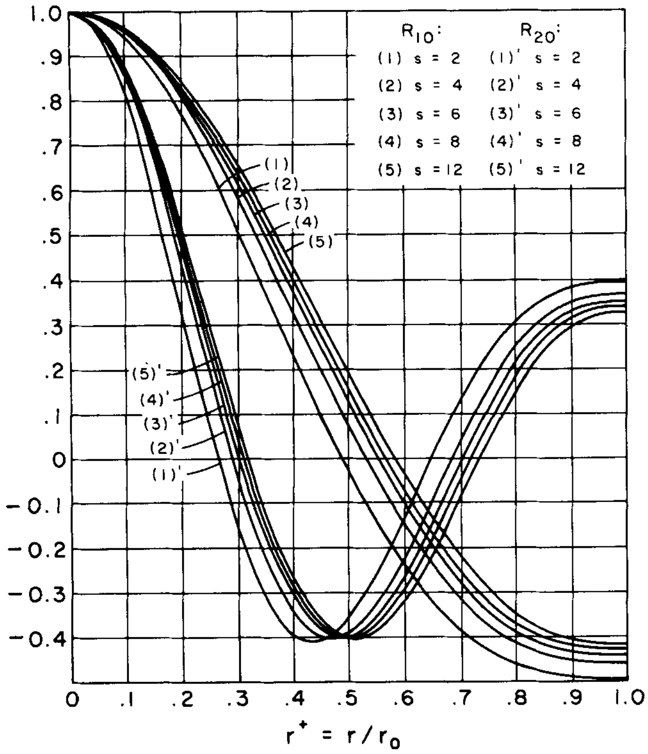

Fig. 2. The first two eigenfunctions for different non-Newtonian fluid behavior index s; $p = 0$.

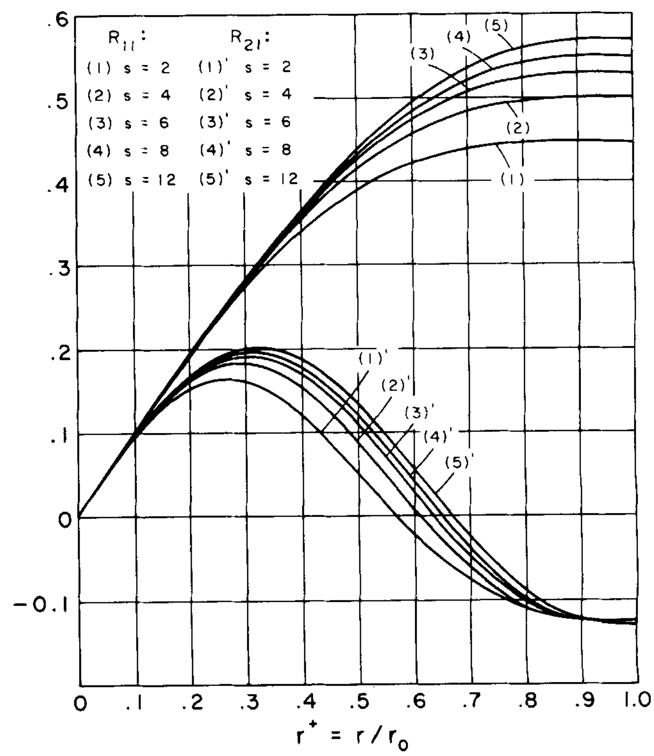

Fig. 3. The first two eigenfunctions for different non-Newtonian behavior index s and for $p = 1$.

Table 2. Comparison of Local Nusselt Numbers for the Circular Tube; Constant Heat Rate; Thermal Entry Length with Kays (1966)

x^+	Kays' (1966) table 8-6 $Nu(x^+)$	Present work $Nu(x^+)$
0.001	Not calculated	15.758
0.002	12.00	12.537
0.004	9.93	9.986
0.010	7.49	7.494
0.020	6.14	6.148
0.040	5.19	5.198
0.10	4.51	4.514
∞	4.36	4.364

Table 3. Local Nusselt Numbers for Laminar Flow of Power Law, Non-Newtonian Fluids in the Thermal Entrance Region of a Circular Pipe with Uniform Wall Heat Flux

x^+	$s = 4$ $Nu(x^+)$	$s = 6$ $Nu(x^+)$	$s = 8$ $Nu(x^+)$	$s = 10$ $Nu(x^+)$	$s = 12$ $Nu(x^+)$
0.001	17.927	19.950	20.947	22.095	23.089
0.002	14.238	15.534	16.583	17.463	18.220
0.004	11.335	12.350	13.163	13.838	14.411
0.01	8.507	9.255	9.842	10.320	10.717
0.02	6.989	7.559	8.068	8.442	8.748
0.04	5.930	6.449	6.838	7.143	7.387
0.1	5.195	5.662	6.003	6.264	6.471
∞	5.053	5.517	5.854	6.109	6.310

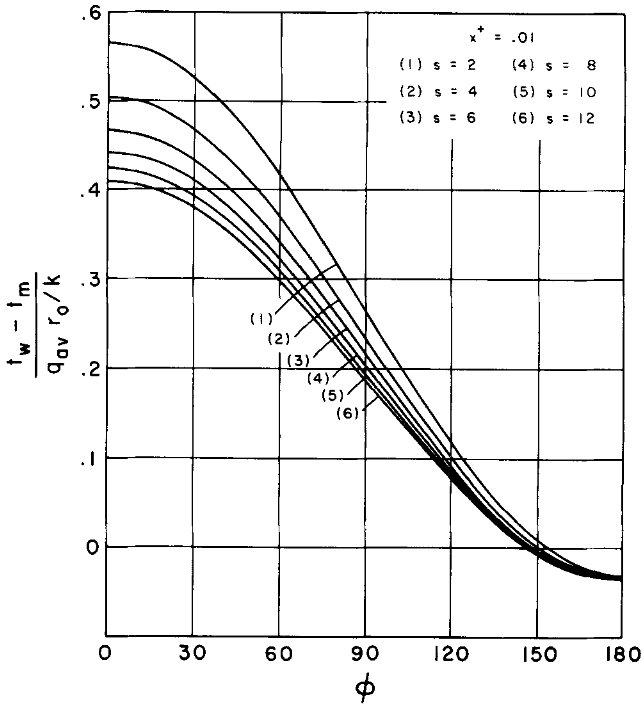

Fig. 4. Illustration of entrance effect of prescribed wall heat flux variation $q(\phi) = q_{av}(1 + \cos\phi)$ and non-Newtonian influence on wall-to-bulk temperature difference at $x^+ = 0.01$.

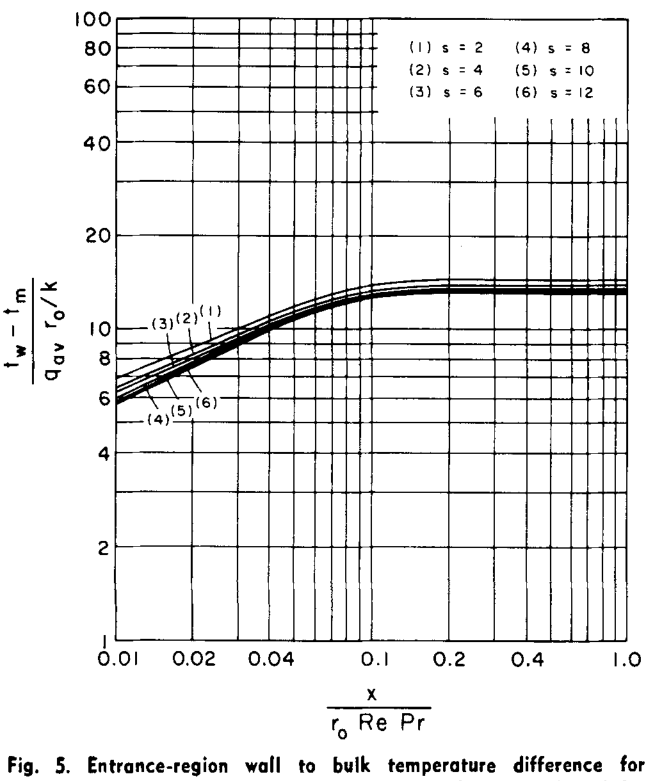

Fig. 5. Entrance-region wall to bulk temperature difference for prescribed wall heat flux variation $q(\phi) = q_{av}(1 + \cos\phi)$ and for different non-Newtonian fluid behavior index s at the angular position $\phi = 0$.

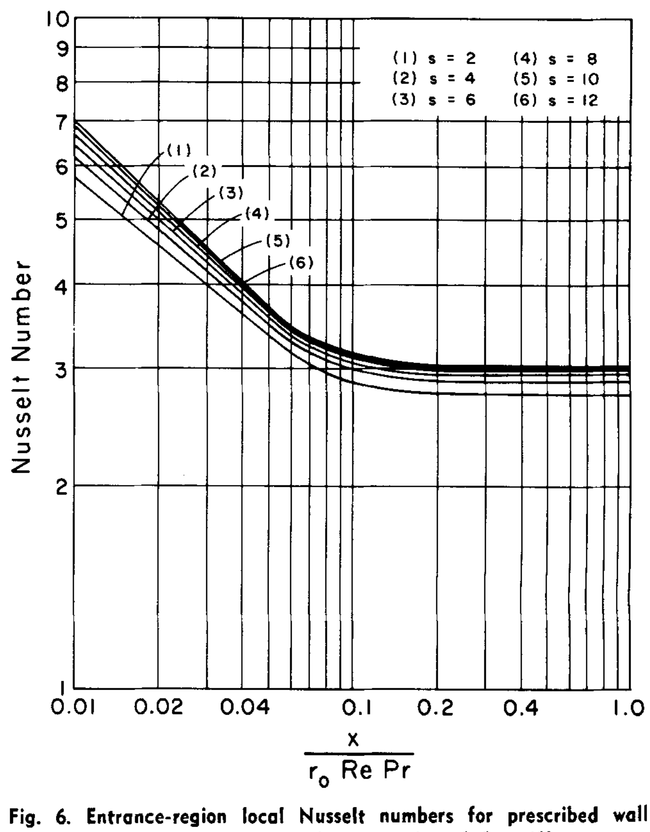

Fig. 6. Entrance-region local Nusselt numbers for prescribed wall heat flux variation $q(\phi) = q_{av}(1 + \cos\phi)$ and for different non-Newtonian fluid behavior index s at the angular position $\phi = 0$.

Dimensionless wall temperature and Nusselt numbers are plotted as functions of axial position for various values of the non-Newtonian parameter s at the location of maximum wall heat flux ($\phi = 0$) in Figures 5 and 6, respectively.

These three figures illustrate certain phenomenological variations. A much more complete set of results is given by Faghri (1974).

In Figure 4 the local wall temperature variation around the tube periphery is observed to be significantly affected by non-Newtonian characteristics of the fluid. The maximum effect occurs at $\phi = 0$, the location of maximum heat flux.

Non-Newtonian effects are seen to influence wall temperature and local Nusselt number only slightly. Values for t_{wall} and Nu do vary with the non-Newtonian index s

at all axial locations. The thermal entry region is observed to extend from the tube entrance to a value of the parameter x^+ approximately equal to 0.2. Non-Newtonian effects apparently do not alter the extent of the thermal entry region.

NOTATION

a_n, b_n = Fourier coefficients
a_{nO}, a_{np}, b_{np} = expansion coefficients
$\hat{a}_{nO}, \hat{a}_{np}$ = defined as $\hat{a}_{nO} = 4a_{nO}$, $\hat{a}_{np} = 4p/b\ a_{np}$
b = heat flux parameter for the special example
h = heat transfer coefficient
n = exponent in the power law model
$Nu(x^+, \phi)$ = local Nusselt number
p = integer parameter in Equation (14)
Pr = Prandtl number
$q(\phi)$ = arbitrary variation of circumferential wall heat flux
$\bar{q}$ = defined by Equation (6d)
r = radial coordinate from the center of pipe
r^+ = dimensionless radial coordinate, r/r_O
r_O = pipe radius
Re = Reynolds number
$R_{np}(r^+)$ = eigenfunctions of the characteristic Equation (14)
s = exponents in the power law model
$t(x, r, \phi)$ = local fluid temperature
$u(r)$ = local fluid velocity
v = average fluid velocity
x = axial coordinate from the inlet point
x^+ = dimensionless axial position,

$$\frac{2s}{s+2}\ \frac{x/r_O}{RePr} \quad \text{or} \quad \frac{2v}{u_{max}}\ \frac{x/r_O}{RePr}$$

Greek Letters

α = thermal diffusivity
θ = local dimensionless fluid temperature
θ^+ = dimensionless entrance region temperature
$\hat{\theta}_{fd}$ = defined by Equation (16)
λ_{np} = eigenvalues of characteristic Equation (14)
ϕ = angular coordinate, deg.
ω = weighting function

Subscripts

av = average value
fd = far away from the entrance
m = evaluated at the mixed mean state
max = maximum value
w = evaluated at wall condition
ϵ = evaluated at the tube entrance

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Growth Rate of an Ice Crystal in Subcooled Pure Water

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New and extensive experimental data on the rate of growth of ice crystals in the *a* axis direction in quiescent and slow flowing subcooled pure water show conclusively that thermal natural convection is an important heat transfer mechanism controlling the growth rate. At zero and low forced velocities, steady growth is observed only when the crystals grow horizontally or upward. Steady downward growth does not occur in quiescent water. This is consistent with the physical properties of water and the phenomenon of thermal natural convection.

Growth rates at high water flow rates vary as the square root of the forced velocity and the 3/2 power of the subcooling and follow the theory of Fernandez and Barduhn (1967) with the ice-water interfacial energy set at 52 mJ/m² (52 erg/cm²).

SCOPE

A new experimental apparatus has been built to measure the growth rate of ice crystals in quiescent and flowing subcooled water. Forced velocities range from 7 × 10^{-5} to 70 cm/s (six orders of magnitude), and growth in quiescent water is also measured. The crystallographic growth direction is in that of the *a* axis. No growth in

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